Abstract

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The Polynomial Formulation of the $U(1)$ Non-Linear σ-Model in 2 Dimensions

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The $SU(N)$ non-linear σ-model [1] dynamics can be described by using a 'polynomial' action which is first order in the derivatives of the fields [2, 3, 4]. The basic idea is that the usual action may also be written in terms of a $(SU(N))$ 1-form field L_μ which satisfies the Maureer-Cartan (flatness) equation $F_{\mu\nu}(L) = 0$ [5]. The polynomial formulation consists in writing the original action in terms of the vector field, adding a Lagrange multiplier term which enforces the Maureer-Cartan equation. As $F_{\mu\nu}$ is a 2-form field (i.e., an antisymmetric tensor), so must be the Lagrange multiplier.

From the usual description based on fields with values on a non-linear group manifold, and with the proper group-invariant path integral measure, one goes to another one in terms of a non-Abelian vector gauge field plus an antisymmetric (2-form) tensor field, both with linear path integral measures [3]. In three and higher spacetime dimensions there appears a gauge symmetry under transformations of the antisymmetric tensor field [6] which, when gauge fixed, requires of course the introduction of the corresponding Faddeev-Popov ghosts. In four dimensions, this symmetry is reducible [7], thus a proper BRST treatment prescribes the introduction of ghosts of ghosts. Note also that the bosonic part of the action is first order in the derivatives, and, depending on the gauge fixing one uses, the ghosts' action may or may not be first order.

The $O(2)$ ($\sim U(1)$) non-linear σ-model in two (Euclidean) dimensions (or $X - Y$ model) is the simplest possible example of application for the polynomial formulation. Besides the non-existence of the gauge symmetry linked to the antisymmetric tensor field $\theta_{\mu\nu}$ in higher dimensions, this field can be decomposed as: $\theta_{\mu\nu} = \epsilon_{\mu\nu} \theta$ where θ is a pseudoscalar field. Despite these simplifications, the model itself is far from trivial, and in its Statistical Mechanics version it undergoes the celebrated Kosterlitz-Thouless phase transition [8]. Indeed, this phase transition occurs precisely because the system has room for the existence of vortices. It is also well known

that it can be mapped to a Sine-Gordon model [9], the correspondence between the $O(2)$ spin field and the Sine-Gordon field being non-local. Thus the question presents itself how does one introduce vortices or any other singular configuration.

The object of this paper is to study the polynomial formulation for the $O(2)$ model in two dimensions, showing how some familiar results of the usual formulation reemerge in a simpler way, and also how some extensions can be implemented. In particular, the Sine-Gordon description is obtained simply by integrating out the vector field and thus deriving an 'effective' action for the Lagrange multiplier, which becomes precisely equal to the Sine-Gordon one.

The structure of this paper is as follows: In section 1 we briefly review the model in its usual formulation, describing some properties which we reformulate in their polynomial form in section 2. In section 3 we construct the mapping to the Sine-Gordon model, and in section 4 we show how to introduce strings of vortices.

1 The non-polynomial or 'second order' formulation

The model is usually defined in terms of the Euclidean action (we follow the presentation of ref. [9])

$$S = \frac{1}{2t} \int d^2x \ \partial_\mu \mathbf{s}(x) \cdot \partial_\mu \mathbf{s}(x) \tag{1}$$

where $\mathbf{s}(x)$ is a two-component, real, continuum spin field

$$\mathbf{s}(x) = (s_1(x), s_2(x)) \ , \ \mathbf{s}^2(x) = 1 \ , \tag{2}$$

and t is a (dimensionless) parameter which plays the role of a temperature.

To solve the constraint on the modulus of $\mathbf{s}$ one can parametrize it as

$$\mathbf{s}(x) = (\cos \phi(x), \sin \phi(x)) \tag{3}$$

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where now ϕ must be pseudoscalar under parity transformations of the two-dimensional spacetime (reflections about one of the axis), and one should note that physical quantities are 2π-periodic functions of ϕ (i.e., it is an angular variable). The $O(2)$ symmetry has been transformed into invariance under rigid translations of ϕ. The partition function is then

$$\mathcal{Z} = \int \mathcal{D}\phi \, \exp[-S(\phi)] \, , \tag{4}$$

where

$$S(\phi) = \int d^2x \, \frac{1}{2t} \, \partial_\mu\phi(x)\partial_\mu\phi(x) \, . \tag{5}$$

Every correlation function of spin variables can be obtained by linear combination of correlation functions of exponentials of the field ϕ

$$\langle \prod_{j=1}^{n} \exp[i\epsilon_j\phi(x_j)] \rangle \, , \, \epsilon_j = \pm 1 \, , \, \forall j \tag{6}$$

where the x_j's are the arguments of the spin fields in the corresponding correlation function. In the spin-wave approximation one neglects singular configurations related to the periodicity of ϕ, and then the correlation function (6) can be exactly calculated as

$$\langle \prod_{j=1}^{n} \exp[i\epsilon_j\phi(x_j)] \rangle = (\frac{\Lambda}{m})^{-\frac{tn}{4\pi}} \prod_{j<k}^{n} (m \mid x_j - x_k \mid)^{\frac{t}{2\pi}\epsilon_j\epsilon_k} \, , \tag{7}$$

where Λ and m are UV and IR cutoffs respectively. When one takes the $m \to 0$ limit, the only non-zero correlation functions are the ones which satisfy the 'neutrality' condition:

$$\sum_{j=1}^{n} \epsilon_j = 0 \, , \tag{8}$$

and this is precisely the condition for $O(2)$-invariance of the correlation functions; i.e., invariance under translations of ϕ. In particular this implies the vanishing of the average of the one-spin function (as prescribed by the Mermin-Wagner theorem [10])

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and, regarding the 2-point correlation function, (7) yields

$$\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle = (\frac{\Lambda}{m})^{-\frac{t}{2\pi}} (m \mid x - y \mid)^{-\frac{t}{2\pi}} . \tag{9}$$

So far the discussion has been confined to the low-temperature phase, characterized by an algebraic decaying of the 2-point correlation function (9). Let us consider now the high-temperature phase, where vortices can appear unconfined, and the periodicity of ϕ is important. The vortices are finite-energy configurations such that the angular field ϕ changes by $2\pi n$, where n is an integer, when one moves around a point, the 'center' of the vortex. n is known as the 'winding' of the vortex. A typical N-vortex configuration, with centers X_j, (j=1, ..., N), and windings n_j is

$$\phi_N(x, \{X_j, n_j\}) = \sum_{j=1}^{N} n_j \, \phi_j(x, X_j)$$

$$\phi_j(x, X_j) = \arctan[\frac{(x - X_j)_2}{(x - X_j)_1}] . \tag{10}$$

The contribution of (10) to the action is then easily evaluated

$$S_N = -\frac{\pi}{t} \sum_{j=1}^{N} n_j^2 \log(\frac{\Lambda}{m}) - \frac{2\pi}{t} \sum_{j<k}^{N} n_j n_k \log \mid X_j - X_k \mid . \tag{11}$$

In the last equation, the neutrality condition for the vortices' charges must also be assumed in order to have a non-zero action when the IR cutoff is removed. When also spin waves are present, the total action becomes the sum of (11) and the usual spin wave part. In what follows the (usual) approximation is made of considering only windings equal to ± 1 for the vortices, since they are the most relevant. Due to this constraint, each sector must contain a fixed number of vortex-antivortex pairs, and we shall include a fugacity η plus a combinatorial (classical) factor to take into account the indistinguishability of the vortices of equal charge. The partition function is then calculated by summing (with the appropriate weight factors) over all the possible configurations within each topological sector, and then over all the

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topological sectors. Sumarising,

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{\eta^{2N}}{(N!)^2} \int \prod_{j=1}^{N} dX_j^2 dY_j^2 \, \mathcal{Z}_N(\{X_j, Y_j\}) , \tag{12}$$

where $\mathcal{Z}_N$ is the partition function for the spin variables in the presence of N vortex-antivortex pairs, with coordinates X_j, Y_j, respectively. One easily realises that the spin-wave contribution factors out, and the total partition function becomes the product of the partition function of a free scalar field by a Coulomb gas partition function. This Coulomb gas partition function can then be mapped to a Sine-Gordon partition function [9]. However, the correspondence between the *fields* in both partition functions is somewhat *ad hoc* in this framework.

2 The polynomial or 'first order' formulation

We know from the previous section that the only non-trivial correlation functions are the $O(2)$-invariant ones, and this is tantamount of invariance under translations in ϕ. It seems then natural to look for the possibility of describing the system completely in $O(2)$ invariant terms. We realize that the simplest possible ϕ-translation invariant field variable is

$$L_\mu(x) = \frac{1}{g} \partial_\mu \phi(x) , \tag{13}$$

where g is a constant with dimensions of mass, introduced to make L_μ dimensionless. We may of course rewrite the action (5) in terms of L_μ only, but we must also take into account the fact that L_μ is not entirely arbitrary, but a 'pure gauge' field, i.e.,

$$\epsilon_{\mu\nu} \partial_\mu L_\nu(x) = 0 . \tag{14}$$

When vortices are present, (14) is relaxed, allowing for a discrete set of points where the rhs is non-zero.

We construct the first order action for the system in the spin wave sector by rewriting (5) in terms of L_μ and then adding a Lagrange multiplier term for the

condition (14):

$$S_{sw} = \int d^2x (\frac{g^2}{2t} L_\mu L_\nu - ig\theta\epsilon_{\mu\nu}\partial_\mu L_\nu) \tag{15}$$

where θ is a scalar field which enforces condition (14). $O(2)$ invariant correlation functions are then calculated in terms of L_μ, observing that [4]

$$\exp\{i[\phi(x_1) - \phi(x_2)]\} = \exp[ig\int_{x_2}^{x_1} dz_\mu L_\mu(z)] \tag{16}$$

where the line integral is taken along any smooth path joining x_2 to x_1. (16) *defines* the lhs in terms of the rhs[1]. In order to give a consistent definition of a local spin field, we should require (16) to be path-independent. This will happen as long as:

$$g\oint_C dz_\mu L_\mu(z) = 2\pi N \tag{17}$$

(where N is any integer) for every closed curve C.

The general non-zero correlation function of the usual formulation will satisfy condition (8), then for each $\epsilon_j = +1$ in (7), there must be an $\epsilon_k = -1$. Whence the general correlation function of the usual formulation can be constructed by forming 'neutral pairs' like the lhs of (16), and then writing them in terms of the corresponding Wilson line on the rhs (of course there are many different ways to chose the pairings, all giving the same result).

The generalization of (14) to the case when N vortex-antivortex pairs are present is just

$$\begin{aligned} \epsilon_{\mu\nu}\partial_\mu L_\nu(x) &= \rho_N(x) \\ \rho_N(x) &= \frac{2\pi}{g}\sum_{j=1}^N [\delta(x - X_j) - \delta(x - Y_j)] \, , \end{aligned} \tag{18}$$

where we use the notation of Equation (12). Within the topological sector defined by (18), the generating functional of L_μ correlation functions is

$$\mathcal{Z}_N(J) = \int \mathcal{D}L_\mu \mathcal{D}\theta \, \exp(-S_N + \int d^2x J_\mu L_\mu) \, , \tag{19}$$

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[1]This is done just to make contact with the usual description, but there is no necessity to introduce the spin fields, neither to solve the constraints on L_μ within the polynomial formulation.

where

$$S_N = \int d^2x \left\{ \frac{g^2}{2t} L_\mu(x) L_\mu(x) - ig\theta \left[\epsilon_{\mu\nu} \partial_\mu L_\nu - \rho_N(x) \right] \right\} \tag{20}$$

with ρ_N as defined by (18). Let us calculate then the spin-spin correlation function in this sector

$$\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle_N = \Re \int \mathcal{D}L_\mu \mathcal{D}\theta \, \exp[i \int_y^x dy_\mu L_\mu(y)] \, \exp(-S_N) \tag{21}$$

(where $\Re$ means real part of). Thus (21) can be obtained from $\mathcal{Z}_N(J)$ just by specifying the current J_μ which reproduces the 'Wilson line' for L_μ, i.e.,

$$J_\mu(u) = ig \int_0^1 ds \, \frac{dz_\mu}{ds} \, \delta(u - z(s)) \,. \tag{22}$$

We first calculate $\mathcal{Z}_N(J)$ for arbitrary J and then we take it to be equal to (22). Integrating L_μ in (19), we get

$$\mathcal{Z}_N(J) = \int \mathcal{D}\theta \, \exp[-\frac{t}{2} \partial_\mu \theta \partial_\mu \theta - \frac{t}{2g^2} J^2 + \theta(-\frac{t}{g} \epsilon_{\mu\nu} \partial_\mu J_\nu + ig\rho_N)] \,. \tag{23}$$

The integral over θ is also Gaussian. The final result is

$$\begin{aligned}
\mathcal{Z}_N(J) &= \exp[-\frac{t}{2g^2} \int d^2x \partial \cdot J \partial^{-2} \partial \cdot J + \frac{g^2}{2t} \int d^2x \rho_N \partial^{-2} \rho_N \\
&\quad - \int d^2x \epsilon_{\mu\nu} \partial_\mu J_\nu \partial^{-2} \rho] \,.
\end{aligned} \tag{24}$$

Using for J_μ the explicit form (22), (24) yields

$$\begin{aligned}
\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle_N &= \exp[-\frac{t}{2\pi} \log \mid x - y \mid] \exp[-\frac{\pi}{t} \sum_{i,j=1}^N \log \mid X_i - Y_j \mid] \\
&\quad \times \cos\{\sum_{k=1}^N [\alpha(x - X_k) - \alpha(x - Y_k) - \alpha(y - X_k) + \alpha(y - Y_k)]\} \\
&\equiv G(x, y; \{X_k, Y_k\}; t) \tag{25}
\end{aligned}$$

where we have absorbed the divergent factors in a renormalization of the fields, and we use the function G to denote explicitly the dependence of the correlation function also on the coordinates of the vortices and the temperature.

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Now we particularize equation (25) for the case $N = 1$

$$
\begin{aligned}
G(x, y; X, Y; t) &= \exp[-\frac{t}{2\pi} \log | x - y |] \exp[-\frac{2\pi}{t} \log | X - Y |] \\
&\times \cos[\alpha(x - X) - \alpha(x - Y) - \alpha(y - X) + \alpha(y - Y)], \quad (26)
\end{aligned}
$$

where $\alpha(x) = \arctan x_2/x_1$ for any $x = (x_1, x_2)$. The correlation function (26) is symmetric under the interchange of spin and vortex coordinates, plus a transformation of the temperature:

$$
x \leftrightarrow X , \; y \leftrightarrow Y \; ; \; t \to \frac{4\pi^2}{t} . \tag{27}
$$

Of course, this symmetry is inherent to the model, and not a consequence of using the polynomial formulation. However, we will need it to realise this symmetry as the result of some invariance of the path integral under a transformation of the *fields*. To achieve this, we need a description where spins field and vortices field appear in a more symmetrical way. L_μ is the spin field and one can realize that θ plays the role of a vortex field. Indeed, the path integral representation of (26) as defined by (21) is

$$
\begin{aligned}
\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle &= \Re \int \mathcal{D}L_\mu \, \mathcal{D}\theta \, \exp[\int d^2x(-\frac{g^2}{2t} L_\mu L_\mu + ig\theta \epsilon_{\mu\nu} \partial_\mu L_\nu)] \\
&\times \exp(ig \int_y^x dz_\mu L_\mu) \exp[-2\pi i(\theta(X) - \theta(Y))] , \tag{28}
\end{aligned}
$$

where one can see that the exponential of θ creates vortices when averaged with the spin wave action. We note that

$$
\begin{aligned}
&\int \mathcal{D}\theta \exp[\int d^2x ig\theta \epsilon_{\mu\nu} \partial_\mu L_\nu] \exp[-2\pi i(\theta(X) - \theta(Y))] \\
&= \int \mathcal{D}\theta_\mu \mathcal{D}\Lambda \exp[\int d^2x(-ig^2 \epsilon_{\mu\nu} L_\mu \theta_\nu + ig\Lambda \epsilon_{\mu\nu} \partial_\mu \theta_\nu)] \\
&\times \exp[-2\pi ig \int_Y^X dz_\mu \theta_\mu(z)] , \tag{29}
\end{aligned}
$$

where Λ is a new Lagrange multiplier field, which enforces the condition $\epsilon_{\mu\nu} \partial_\mu \theta_\nu = 0$, which is solved by $\theta_\mu(x) = \frac{1}{g} \partial_\mu \theta(x)$. Inserting (29) into (28) we get a more

symmetrical description, in terms of the spin field L_μ and vortex field θ_μ,

$$
\begin{aligned}
\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle &= \Re \int \mathcal{D}L_\mu \, \mathcal{D}\theta_\mu \, \mathcal{D}\Lambda \\
&\times \exp\Big[\int d^2x\big(-\frac{g^2}{2t}L_\mu L_\mu - ig^2\epsilon_{\mu\nu}\theta_\mu L_\nu + ig\Lambda\epsilon_{\mu\nu}\partial_\mu\theta_\nu\big)\Big] \\
&\times \exp\big(ig\int_y^x dz_\mu L_\mu(z)\big) \exp\big[-2\pi ig\int_Y^X dz_\mu\theta_\mu(z)\big] .
\end{aligned}
\tag{30}
$$

Thus, the spin-spin correlation function corresponds to the average of the product of two 'Wilson lines', one for L_μ and the other for θ_μ:

$$
\langle \mathbf{s}(x) \cdot \mathbf{s}(y) \rangle = \Re\langle\exp[ig\int_y^x dz \cdot L] \exp[2\pi ig\int_Y^X dz \cdot \theta]\rangle ,
\tag{31}
$$

where the average is performed with the action

$$
S = \int d^2x\{\frac{g^2}{2t}L^2 + ig^2\epsilon_{\mu\nu}\theta_\mu L_\nu - ig\Lambda\epsilon_{\mu\nu}\partial_\mu\theta_\nu\} .
\tag{32}
$$

Then, the duality transformation amounts to performing the following transformation in thew action (32):

$$
L_\mu(x) \to 2\pi\theta_\mu(x) , \ \ \theta_\mu(x) \to \frac{1}{2\pi}L_\mu(x) , \ \ t \to \frac{4\pi^2}{t} .
\tag{33}
$$

One easily verifies that the average of the same Wilson's lines operators with the transformed action gives the transformation (27). Note that the effect of this change of variables is to interchange the roles of vortices and spins, as well as low and high temperatures. In this sense, $t = \frac{4\pi^2}{t}$ is a kind of self-dual point, where spins and vortices are interchangeable.

3 Mapping to the Sine-Gordon Model.

The total partition function is defined following (12)

$$
\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{(N!)^2}\eta^{2N} \int \prod_{j=1}^{N} d^2X_j d^2Y_j \mathcal{Z}_N(0) ,
\tag{34}
$$

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with

$$\mathcal{Z}_N(0)(\{X_j, Y_j\}) = \int \mathcal{D}L_\mu \mathcal{D}\theta \ \exp\{-\int d^2x[\frac{g^2}{2t}L_\mu L_\mu - ig\theta(\epsilon_{\mu\nu}\partial_\mu L_\nu - \rho_N)]\} ,$$

$$\rho_N(x) = \frac{2\pi}{g}\sum_{j=1}^N [\delta(x - X_j) - \delta(x - Y_j)] . \tag{35}$$

Performing the Gaussian integration over L_μ in (35), and using the delta functions in the definition of ρ_N, we can rewrite $\mathcal{Z}_N$ as

$$\mathcal{Z}_N = \langle \prod_{j=1}^N \exp[2\pi i(\theta(X_j) - \theta(Y_j))] \rangle , \tag{36}$$

where the average $\langle\rangle$ is defined by

$$\langle F(\theta) \rangle = \int \mathcal{D}\theta F(\theta) \ \exp[-S(\theta)] ,$$

$$S(\theta) = \frac{t}{2}\int d^2x \partial_\mu\theta\partial_\mu\theta . \tag{37}$$

Using (37) in (34), we can rewrite the total partition function $\mathcal{Z}$ as

$$\mathcal{Z} = \sum_{N=0}^\infty \frac{\eta^{2N}}{(N!)^2}\langle [\int d^2x e^{-2\pi i\theta(x)}]^N [\int d^2y e^{+2\pi i\theta(y)}]^N \rangle . \tag{38}$$

Taking now into account the fact that only products of exponentials that satisfy the 'neutrality' condition are non-zero, it is straightforward to check that

$$\mathcal{Z} = \langle \exp(2\eta \int d^2x \cos(2\pi\theta)) \rangle \tag{39}$$

or

$$\mathcal{Z} = \int \mathcal{D}\theta \ \exp\{-\int d^2x[\frac{t}{2}(\partial\theta)^2 - 2\eta \cos(2\pi\theta)]\} \tag{40}$$

which is the desired Sine-Gordon action in terms of the Lagrange multiplier field θ. This coincides with the result obtained by more traditional methods.

The effective action for θ in (40) was obtained under the assumption that the vortices can only have charges equal to ± 1. Let us consider now an extension of the model. It consists of relaxing this constraint, allowing for the charges to be equal

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to plus or minus any real number q. Of course, the local spin interpretation will no longer be true, since the definition (16) requires condition (17). In this sense this can be considered as an extension of the $O(2)$ model, which allows for vortices of non-integer charge. However, we will first assume a unique value for q in the model. The topological sectors will then have $2N$ vortex-antivortex pairs as before. Again only neutral combinations will have a finite weight. Of course we will also average over the positions of the vortices, and include the corresponding combinatorial factors. One can verify that instead of obtaining the Sine-Gordon partition function (40), we get

$$\mathcal{Z} \;=\; \int \mathcal{D}\theta \, \exp\{-\int d^2x [\frac{t}{2}(\partial\theta)^2 - 2\eta \cos(2\pi q\theta)]\} \;. \tag{41}$$

Thus, even when the change performed on the spin system is drastic, the Sine-Gordon parameters change smoothly from $q = 1$ to any q. Things change more dramatically if we average now over all the possible values of q, since this produces delta-functions of the field θ. The result is a kind of field-theoretic delta-function model:

$$\mathcal{Z} \;=\; \int \mathcal{D}\theta \, \exp\{-\int d^2x [\frac{t}{2}(\partial\theta)^2 - \eta\delta(\theta)]\} \;. \tag{42}$$

4 Strings of vortices

We have shown how the usual point-like singularities (vortices) are introduced in the polynomial formulation. Let us consider now string-like singularities (which could be regarded as strings of vortices). The obvious generalisation of the procedure we followed for the point-like case is to impose on L_μ a constraint like:

$$\epsilon_{\mu\nu}\partial_\mu L_\nu(x) \;=\; \rho_{st}(x) \;, \tag{43}$$

where ρ_{st} is the density appropriate to a string:

$$\rho_{st}(x) \;=\; \frac{2\pi}{g} \int_0^1 ds \, q(s) \, \delta[x - \gamma(s)] \tag{44}$$

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where the string's path is parametrized by $\gamma(s) : [0,1] \to \mathrm{R}^2$, and $q(s)$ measures the density of vorticity along the curve. Of course we can consider more than one string, just by using in the rhs of (44) the sum of the densities corresponding to the paths. One can see that the spin configuration which corresponds to a state like the one defined by (43) and (44) is:

$$
\begin{aligned}
\mathbf{s}(x) &= (\cos\phi(x), \sin\phi(x)) , \\
\phi(x) &= \int_0^1 ds\, q(s) \arctan(x - \gamma(s)) .
\end{aligned} \tag{45}
$$

As for the vortices, the non-local definition of the spin field will be consistent only if condition (17) is met for any curve[2]. When strings are present this implies:

$$
\int_0^1 ds\, q_j(s) = N_j , \quad \forall j , \tag{46}
$$

where the N_j's are integers. Also, in order for the action to be non-zero when the IR cut-off is removed, a neutrality condition must be satisfied

$$
\sum_{j=1}^N \int_0^1 q_j(s) = 0 . \tag{47}
$$

When only one string is present (47) supersedes (46), and thus there is not true singularity since (45) implies that the net rotation of the spin in the local frame is null. Note that there is a crucial difference between open and closed strings, because while in the former the Wilson line definition works globally, in the latter it applies only to one of the simply-connected regions into which the space becomes divided (we recall that we cannot cross a singularity with the Wilson line). Let us estimate the Boltzmann weight of a configuration of a N string-antistring configuration. It is straightforward to calculate the action due to this configuration in the polynomial formulation. It becomes just the Coulomb energy of the corresponding charge distribution:

$$
S_{st} = \frac{\pi}{t} \sum_{i,j=1}^N \int_0^1 ds_1 \int_0^1 ds_2\, q_i(s_1)\, q_j(s_2)\, \log | \alpha_i(s_1) - \beta_j(s_2) | , \tag{48}
$$

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[2]We do not allow the curves to intercept any singularity.

where α and β parametrize the strings and antistrings paths, respectively (we have not written the (divergent) self-energies). With the natural definition of the mean position (X_j) of the j-th string,

$$\log(x - X_j) \; = \; \int_0^1 ds q_j(s) \log(x - \alpha(s)) \;, \; (\int_0^1 ds q_j(s) = +1) \;, \tag{49}$$

and analogously for an antistring, the action looks exactly like the one of N vortex-antivortex pairs, i.e.,

$$S_{st} \; = \; \frac{\pi}{t} \sum_{j,k=1}^N \log \mid X_j - Y_k \mid \; . \tag{50}$$

Thus we have arrived at the conclusion that the Botzmann weight of these configurations is like the one of the usual point-like vortices. However, large strings are suppressed in the partition function because of the restriction about their positions. For example, the volume inside a closed string cannot be occupied by another one, and so they should be more strongly suppressed than the open ones.

Acknowledgements

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